This week

1. Directives concerning the MLP test
2. Section 3.1: tangents and the derivative at a point
3. Section 3.2: the derivative as a function
4. Section 3.4: velocity
Tests with MyLabsPlus

- All tests will take place in Therm, using dedicated ‘ChromeBook’ laptops.
- The test will not be available until 8:45 sharp.
- You have 60 minutes to complete the test (75 minutes for dislectic students).
- Be there well in time. If you use public transportation, take one bus or train earlier than usual. Although you can start late (but not later than 9:15), this should be an exception. Be well aware that you will disturb your fellow students that already started their test.
- The use of an electronic calculator (or any other device) is not allowed. A calculator will be available on the chromebook as a separate app.
- A trigonometry formula sheet will be issued before the test. This formula sheet can be reviewed on Blackboard.
- A practice test is available.

Tests with MyLabsPlus

- Elaborate the test questions on paper before you submit an answer to MyLabsPlus.
- Put your name and student number on the paper and hand it in after you completed the test.
- The paper is not graded, but can be used as evidence for reviewing purposes.
- You can review your test after 12:00.
- Distribution of points per test:
  - test 1: 8 points
  - test 2: 12 points
  - test 3: 10 points
- Your mark is the total amount of points divided by 3, with one-decimal precision, and with minimum 1.
Wrong answer means 0 points!

- Use exact answers, avoid decimals if possible:
  
  \[ \frac{3}{4} \rightarrow \frac{1}{4} \]

- If decimals are required, use a decimal point:
  
  \[ 3.14 \rightarrow 3.14 \]

- Simplify your answers as much as possible:
  
  \[ \frac{2}{5} \rightarrow \frac{3}{5} \quad \frac{3}{2} \rightarrow \frac{7}{2} \]

  \[ \frac{6x}{3x^2} \rightarrow 2x \]

  \[ \frac{1}{\sqrt{4}} \rightarrow 1 \quad \frac{1}{\sqrt{3}} \rightarrow \frac{1}{3} \sqrt{3} \]

- Use the right variable. If \( f \) is a function of \( t \), i.e. \( f(t) = t^2 \), then
  
  \[ f'(t) \rightarrow f'(t) = 2t \]

The (angle of) inclination is the angle \( \theta \) that \( \ell \) makes with the \( x \)-axis.

- The angle is measured from the positive \( x \)-axis to \( \ell \).
- Turning counterclockwise means \( \theta > 0 \).
- Turning clockwise means \( \theta < 0 \).
The slope of a line

The slope of \( \ell \) is defined as \( \tan \theta = \frac{\Delta y}{\Delta x} \).

This holds for every choice \( P_1 \) and \( P_2 \), as long as \( P_1 \neq P_2 \).
Let \( \ell \) be the line through \( P = (x_0, y_0) \) with slope \( m \), then for every point \((x, y) \neq P\) on \( \ell \) we have
\[
\begin{align*}
    m &= \frac{y - y_0}{x - x_0} \\
    y - y_0 &= m(x - x_0) \\
    y &= m(x - x_0) + y_0.
\end{align*}
\]

The equation of the line through \( P \) and with slope \( m \) is
\[
y = m(x - x_0) + y_0.
\]

Let \( \ell \) be the line through with slope \( m \) and with \( y \)-intercept \( b \), then \( \ell \) passes through \((0, b)\).

The equation of \( \ell \) is
\[
y = m(x - 0) + b = mx + b.
\]
Equation of a line passing through two points

1.6

- Let \( \ell \) be the line through \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) where \( P_1 \neq P_2 \), then the slope of \( \ell \) is
  \[
m = \frac{y_2 - y_1}{x_2 - x_1}.
\]

- The equation of the line through \( P_1 \) and \( P_2 \) is
  \[
y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.
\]

- Example: the equation of the line through \((1, -1)\) and \((3, 5)\) is
  \[
y = \frac{5 - (-1)}{3 - 1}(x - 1) + (-1),
  
y = 3(x - 1) - 1,
  
y = 3x - 4.
\]

The general equation of a line

1.7

- The general equation of a line is
  \[ax + by = c,\]
  with \( a, b \) and \( c \) real constants, where \( a \) and \( b \) are not both 0.

- If \( b \neq 0 \), then
  \[ax + by = c,\]
  \[by = -ax + c,\]
  \[y = -\frac{a}{b}x + \frac{c}{b},\]
  so the slope is \(-\frac{a}{b}\), and the \( y \)-intercept is \(\frac{c}{b}\).

- If \( b = 0 \), then \( a \neq 0 \), and \( x = \frac{c}{a} \) is the equation of a \textbf{vertical line}.  

- The horizontal line with $y$-intercept $b$ has slope 0 and therefore is described by the equation $y = b$.
- The vertical line with $x$-intercept $a$ has slope $\infty$ and is described by the equation $x = a$. 

**Exercises**

Assignment: IMM1 - Tutorial 3.1
The derivative of a function $y = f(x)$

We define the **derivative** $f(x)$ at $x_0$ as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$ 

The number $f'(x_0)$ can be interpreted as:

- the slope of the graph of $y = f(x)$ at the point $(x_0, f(x_0))$;
- the slope of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$;
- the rate of change of $f(x)$ at the point $x_0$. 

Differentiation - Secant.nb
Example: the derivative of $f(x) = x^2$ at 1

For $f(x) = x^2$ we have

\[
\begin{array}{|c|c|c|c|c|}
\hline
h & 1 + h & f(1) & f(1+h) & \frac{f(1+h) - f(1)}{h} \\
\hline
1 & 2 & 1 & 4 & 3 \\
0.5 & 1.5 & 1 & 2.25 & 1.25 \\
0.25 & 1.25 & 1 & 1.5625 & 0.5625 \\
0.01 & 1.01 & 1 & 1.0201 & 0.0201 \\
0.001 & 1.001 & 1 & 1.002001 & 0.002001 \\
\hline
\end{array}
\]

This suggests: when $h$ approaches 0, then $\frac{f(1+h) - f(1)}{h}$ approaches 2.

Example: the derivative of $f(x) = x^2$ at 1

\[
f(1) = 1^2 = 1
\]

\[
\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1}{h} = \frac{x + 2h + h^2 - x}{h} = \frac{2h + h^2}{h} = 2 + h
\]

\[
f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 2.
\]
Example: the tangent line of \( f(x) = x^2 \) at \((1, 1)\)

- The tangent line has slope \( f'(1) = 2 \) and passes through \((1, f(1)) = (1, 1)\), hence the tangent line is described by the equation \( y = 2 \cdot (x - 1) + 1 = 2x - 1 \).

Example: the derivative of \( f(x) = x^2 \) at \( a \)

\[
f(x) = x^2.
\]

\[
\frac{f(a + h) - f(a)}{h} = \frac{(a + h)^2 - a^2}{h}
\]

\[
= \frac{a^2 + 2ah + h^2 - a^2}{h}
\]

\[
= \frac{2ah + h^2}{h}
\]

\[
= 2a + h.
\]

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = 2a.
\]
Example: the derivative of \( f(x) = \sqrt{x} \) at \( a \)

\[
f(a) = \sqrt{a}.
\]

\[
\frac{f(a + h) - f(a)}{h} = \frac{\sqrt{a + h} - \sqrt{a}}{h}
\]

\[
= \frac{(\sqrt{a + h} - \sqrt{a})(\sqrt{a + h} + \sqrt{a})}{h(\sqrt{a + h} + \sqrt{a})}
\]

\[
= \frac{(a + h) - a}{h(\sqrt{a + h} + \sqrt{a})}
\]

\[
= \frac{h}{h(\sqrt{a + h} + \sqrt{a})}
\]

\[
= \frac{1}{\sqrt{a + h} + \sqrt{a}}.
\]

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \frac{1}{2\sqrt{a}}.
\]

---

Example: the derivative of \( f(x) = 1/x \) at \( a \neq 0 \)

\[
f(a) = \frac{1}{a} \quad (a \neq 0).
\]

\[
\frac{f(a + h) - f(a)}{h} = \frac{1}{a + h} - \frac{1}{a} = \frac{1}{h} \left( \frac{1}{a + h} - \frac{1}{a} \right)
\]

\[
= \frac{1}{h} \left( \frac{a}{a(a + h)} - \frac{a + h}{a(a + h)} \right)
\]

\[
= \frac{1}{h} \left( \frac{a - (a + h)}{a(a + h)} \right) = \frac{1}{h} \left( \frac{-h}{a(a + h)} \right)
\]

\[
= -\frac{1}{a(a + h)}.
\]

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = -\frac{1}{a^2}.
\]

---
(1) Show that $b^3 - a^3 = (b - a)(b^2 + ab + a^2)$.

(2) Let $f(x) = x^3$. Use the definition of the derivative to show that for all real numbers $a$ the following holds:

$$f'(a) = 3a^2.$$
Example: the derivative of \( f(x) = x^2 \)

\[ f'(x) = 2x \]

We already evaluated the derivative of \( f \) at \( a \):

\[ f'(a) = 2a. \]

Replace \( a \) by \( x \): the derivative of \( f \) is the function \( f'(x) = 2x \).

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Example: the derivative of \( f(x) = \sqrt{x} \)

\[ f'(x) = \frac{1}{2\sqrt{x}} \]

The derivative of \( f \) at \( a \) is \( f'(a) = \frac{1}{2\sqrt{a}} \).

Replace \( a \) by \( x \): the derivative of \( f \) is the function \( f'(x) = \frac{1}{2\sqrt{x}} \) \((x > 0)\).
Example: the derivative of $f(x) = 1/x$

![Graph of $f(x) = 1/x$ and $f'(x) = -1/x^2$.]

The derivative of $f$ at $a$ is $f'(a) = -\frac{1}{a^2}$.

Replace $a$ by $x$: the derivative of $f$ is the function $f'(x) = -\frac{1}{x^2}$ $(x \neq 0)$.

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Warning: derivatives are not always defined!

![Graph of $f(x) = \begin{cases} x + 1 & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$ does not have a derivative at $x = 0$.]

The graph of $f(x) = \begin{cases} x + 1 & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$ does not have a derivative at $x = 0$.

- A derivative does not exist at a point where the graph is discontinuous.
The graph of $y = f(x) = |x|$ does not have a derivative at $x = 0$.

- A derivative does not exist at a point where the graph has a sharp spike (called a salient).

The graph of $y = f(x) = \sqrt[3]{x}$ does not have a derivative at $x = 0$.

- A derivative does not exist at a point where the graph has a vertical tangent.
The function \( f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \) is differentiable at 0.

- Piecewise defined functions do not always pose problems.

**Integer powers of \( x \)**

**Recursion Formula**

For all \( n \geq 1 \) we have

\[
\frac{d}{dx}(x^{n+1}) = x \frac{d}{dx}(x^n) + x^n
\]

\[
\frac{d}{dx}(x^{n+1}) = \lim_{h \to 0} \frac{(x + h)^{n+1} - x^{n+1}}{h}
\]

\[
= \lim_{h \to 0} \frac{(x + h)(x + h)^n - x \cdot x^n}{h}
\]

\[
= \lim_{h \to 0} \frac{x(x + h)^n - x \cdot x^n + h(x + h)^n}{h}
\]

\[
= \lim_{h \to 0} x \frac{(x + h)^n - x^n}{h} + \lim_{h \to 0}(x + h)^n
\]

\[
= x \frac{d}{dx}(x^n) + x^n.
\]
Integer powers of $x$

- Let $n = 1$: \( \frac{d}{dx}(x^2) = x \frac{d}{dx}(x) + x = x \cdot 1 + x = 2x. \)
- Let $n = 2$: \( \frac{d}{dx}(x^3) = x \frac{d}{dx}(x^2) + x^2 = x \cdot 2x + x^2 = 3x^2. \)
- Let $n = 3$: \( \frac{d}{dx}(x^4) = x \frac{d}{dx}(x^3) + x^3 = x \cdot 3x^2 + x^3 = 4x^3. \)
- Let $n = 4$: \( \frac{d}{dx}(x^5) = \ldots \)
- Let $n = 5$: \( \frac{d}{dx}(x^6) = \ldots \)

**Theorem**

For all $n \geq 1$ we have \( \frac{d}{dx}(x^n) = n x^{n-1} \)

Powers of $x$

**Theorem**

For all real numbers $\alpha$ we have \( \frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1} \)

**Check:**

- Let $\alpha = \frac{1}{2}$, then
  \[ \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1}. \]
- Let $\alpha = -1$, then
  \[ \frac{d}{dx}(x^{-1}) = \frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2} = -x^{-2} = -1 x^{(-1)-1}. \]
Exercises

- Define \( f(x) = mx + b \). Find the derivative \( f' \) and draw a plot containing the graphs of \( f \) and \( f' \) for the following values of \( m \) and \( b \):

1. \( m = 0.5, \ b = 1 \),
2. \( m = 1, \ b = 0 \),
3. \( m = 2, \ b = -1 \).

Assignment: IMM1 - Tutorial 3.3

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Average velocity

Consider a moving object and assume that we know the traveled distance as a function of time \( s(t) \).

- If the object moves from \( s(t_A) \) to \( s(t_B) \), the displacement is \( s(t_B) - s(t_A) \).
- The average velocity over the interval \( (t_A, t_B) \) is the displacement per elapsed time.

\[
\text{average velocity} = \frac{s(t_B) - s(t_A)}{t_B - t_A}.
\]
Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.

- The **velocity at time** $t_A$ is the limit of the average velocity over the interval $(t_A, t_B)$ where $t_B$ approaches $t_A$:

$$v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}.$$

Define $h = t_B - t_A$, then

- $t_B = t_A + h$ and
- “$t_B \to t_A$” is equivalent to “$h \to 0$”.

$$v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A} = \lim_{h \to 0} \frac{s(t_A + h) - s(t_A)}{h} = s'(t_A).$$

**Velocity is the derivative of displacement.**
Example: the motion of a rocket

4.5

Question: when did the rocket reach its highest point (apex)?

Answer: at $t \approx 8$ seconds.
Example: the motion of a rocket

Question: for how many seconds did the engine burn?

Answer: 2 seconds.

Example: the motion of a rocket

Question: when did the parachute open?

Answer: at $t = 10$ seconds.
Example: the motion of a rocket

Question: what happens here?

Answer: after approximately 12 seconds the rocket reaches terminal velocity, which it keeps for about 8 seconds.

Example: the motion of a rocket

Question: what happens here?

Answer: the rocket hits the ground at $t \approx 20$ seconds.
Example: the motion of a rocket

\[ t \text{ (sec)} \]

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Question: what is the physical interpretation of the second derivative?

Answer: acceleration

Which is better: falling or crashing?

- The acceleration of gravity is \( g = 9.81 \text{ m/s}^2 \).
- The velocity at time \( t \) is \( v(t) = gt \text{ m/s} \).
- The distance travelled is \( s(t) = \frac{1}{2}gt^2 \text{ m} \).
- The fall time \( t_0 \) is found by solving \( s(t_0) = 10 \Rightarrow t_0 = 1.43 \text{ sec} \).
- The velocity when hitting the ground is

\[
v(t_0) = 14.0 \text{ m/s} = \frac{14.0}{1000} \times 3600 \approx 50.4 \text{ km/h}.
\]
Capacitors

Physical principles

(1) In a capacitor, the charge $Q$ on the plates is proportional to the voltage $V$ over the plates: hence $Q = CV$, where $C$ is the capacity.

(2) The current through a lead is the amount of charge per second flowing through the lead.

- From principle (1) we derive:
  $$Q(t + \Delta t) - Q(t) = CV(t + \Delta t) - CV(t) = C\left(V(t + \Delta t) - V(t)\right)$$

- From principle (2) we derive:
  $$Q(t + \Delta t) = Q(t) + I(t)\Delta t \Rightarrow Q(t + \Delta t) - Q(t) = I(t)\Delta t$$

- Combining these results gives: $I(t) = \frac{C}{\Delta t}\frac{V(t + \Delta t) - V(t)}{\Delta t}$

- After taking the limit $\Delta t \to 0$ we see that $I(t) = CV'(t)$

Exercises

Assignment: IMM1 - Tutorial 3.4